A Novel Scheme of Nonfragile Controller Design for Periodic Piecewise LTV Systems

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***Abstract*—In this article, a novel nonfragile controller design scheme is developed for a class of periodic piecewise systems with linear time-varying subsystems. Two types of norm-bounded controller perturbations, including additive and multiplicative ones, are considered and partially characterized by periodic piecewise time-varying parameters. Using a new matrix polynomial lemma, the problems of nonfragile controller synthesis for periodic piecewise time-varying systems (PPTVSs) are made amenable to convex optimization based on the favorable property of a class of matrix polynomials. Depending on selectable divisions of subintervals, sufficient conditions of the stability and nonfragile controller design are proposed for PPTVSs. Case studies based on a multi-input multi-output PPTVS and a mass-spring-damper system show that the proposed control schemes can effectively guarantee the close-loop stability and accelerate the convergence under controller perturbations, with more flexible periodic time-varying controller gains than those obtained by the existing methods.**

***Index Terms*—Matrix polynomial, nonfragile control, periodic systems, time-varying systems.**

I. Introduction

PERIODIC characteristics, models, and tasks are present or required in a multitude of fields, including but not limited to aerospace, mechatronics, networks, and signal processing [1]–[5]. Among them, linear periodic systems play important roles, given their convenience in tackling linear time-varying (LTV) systems similarly to linear time-invariant (LTI) ones [6]. Aimed at the stability and robust performance of linear periodic systems, research efforts have been focused on the Floquet–Lyapunov theory for periodic differential equations [7] and lifting-techniques based discrete-time periodic applications [8] over the past decades. Recent studies [9], [10] have revealed the efficiency of periodic piecewise models as approximations of continuous-time periodic systems that do not necessarily have closed-form expressions. Unlike the numerical computational approaches such as the Floquet–Lyapunov transformation and the monodromy matrix, periodic piecewise systems are less complicated and capable of overcoming the difficulties in controller synthesis brought by continuous-time periodic dynamics, making the related problems more amenable to convex optimization tools.

The investigations on periodic piecewise systems are incipiently based on the model formulation consisting of several LTI subsystems, namely, the periodic piecewise linear system (PPLS). Motivated by the decomposition of periodic dynamics [11] and the theory of switched systems [12], the stability analysis and stabilizing controller design of PPLSs have been achieved by using Lyapunov functions with periodic piecewise matrices [9]. The periodic control scheme for PPLSs has been improved in [13] to provide time-varying controller gains under the framework of finite-time stability. Furthermore, H_∞ control and guaranteed cost control schemes are established for time-delay PPLSs as well as delay-free PPLSs in [14]–[16]. A peak-to-peak filter is designed for PPLSs with polytopic uncertainties in [17]. In [18], a matrix polynomial-based time-varying controller is proposed for PPLSs. For PPLSs with positivity, the stability and L_1-gain are analyzed in [19].

Despite the simplicity of PPLSs, the time-invariant subsystem formulations may result in a loss of system dynamics. In practice, it is preferable to use time-varying models to represent practical systems, such as mechanical systems with periodic time-varying stiffness, loads or motions, and process plants involving periodic variables [20], [21]. Hence, the model of periodic piecewise time-varying system (PPTVS) is recommended. Compared with PPLSs, PPTVSs consist of a number of time-varying subsystems, leading to approximations that may be more desirable to preserve periodic dynamics. On the other hand, the time-varying dynamics in PPTVSs also lead to nonconvex conditions especially during controller synthesis, bringing more difficulties for complicated cases with uncertainties or perturbations. For many engineering applications, controller perturbations commonly exist and result in the degradation of control performance [22], giving rise to the need of nonfragile control schemes, which have received extensive attention in the related fields including multivariable systems [23], switched systems [24], [25], time-delay systems [26], [27], sampled-data

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control [28], [29], and event-triggered control [30]. In [31], the basic issues of stability and control are studied for PPTVSs from a polynomial perspective. In [32], the state tracking controller of PPTVSs is developed. In [33], the standard nonfragile control problem is investigated for PPTVSs affected by constant time delay. However, the previous studies on PPTVSs are with system and controller structures sharing the same widths of the subintervals. In other words, the system and the controller have identical time-varying coefficients, which technically reduced the flexibility in controller design. If the system and the controller are constructed over subintervals with different widths, nonidentical time-varying coefficients will be incurred, making the controller synthesis difficult using the existing methods. In addition, little efforts have been made on nonfragile controller synthesis with uncertain periodic piecewise time-varying perturbations, which motivated this study from a different perspective.

In this article, two types of nonfragile periodic controllers are established to deal with norm-bounded additive and multiplicative perturbations, respectively. A novel lemma on the negative definiteness of a class of matrix polynomials is proposed, providing a flexible scheme of controller design that allows time segmentation with nonidentical time-varying coefficients. Hence, the controller gains can be obtained based on some selectable parameters of subinterval division, which differs our work from the existing results. The contributions and novelties are threefold

1) Novel nonfragile time-varying controllers are developed to resist the controller perturbations partially described by periodic piecewise time-varying functions.
2) A new matrix polynomial lemma is proposed to avoid the coupling terms incurred by time-varying dynamics, providing more technical flexibility in controller design with nonidentical time-varying coefficients.
3) The periodic time-varying controller gains can be efficiently determined by some selected divisions of PPTVS subintervals and solved via convex optimization.

The article is organized as follows. Section II gives the problem formulation. Section III provides the close-loop stability analysis and nonfragile controller synthesis. The effectiveness of the designed control schemes are verified through case studies in Section IV. Section V concludes this article.

Notation: $\mathbb{R}^n$ stands for the n-dimensional Euclidean space; $\mathbb{N}^+$ denotes the set of positive integers; $\|\cdot\|$ denotes the Euclidean norm of a vector; I and 0 represent the identity matrix and zero matrix, respectively. $P>0(\geq 0)$ denotes that P is a real symmetric and positive definite (semidefinite) matrix. P^T and P^{-1}, respectively, denote the transpose and the inverse of matrix P, $\mathbf{sym}(P)=P^T+P$. $\mathbf{diag}(\cdot)$ denotes a diagonal matrix constructed by the given diagonal elements. $\overline{\lambda}(\cdot)$, $\underline{\lambda}(\cdot)$ refer to the maximum, minimum eigenvalues of a real symmetric matrix. In block symmetric matrices, "$*$" is used as an ellipsis for the terms introduced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

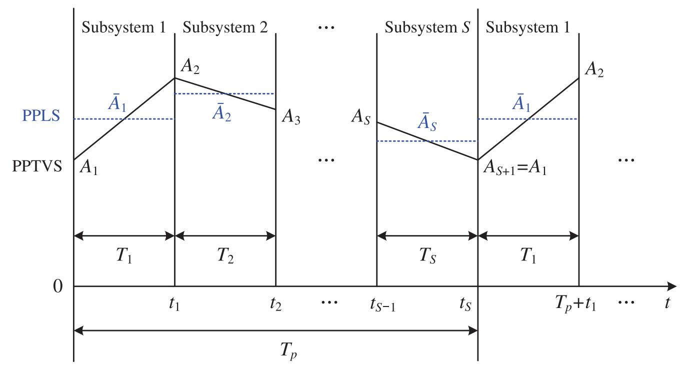

Fig. 1. Evolutions of subsystem matrices in PPTVS (1) and a PPLS.

II. Problem Formulation

Consider a continuous-time PPTVS

$$\dot{x}(t)=\mathcal{A}(t)x(t)+\mathcal{B}(t)u(t) \tag{1}$$

where $x(t)\in\mathbb{R}^n$ and $u(t)\in\mathbb{R}^{n_u}$ are the state vector and control input, respectively; T_p is the fundamental period of system (1) such that $\mathcal{A}(t)=\mathcal{A}(t+lT_p)$ and $\mathcal{B}(t)=\mathcal{B}(t+lT_p)$ for $t\geq 0$, $l=0,1,\ldots$. Time interval $[lT_p,(l+1)T_p)$ is supposed to be partitioned into S subintervals, namely $[lT_p+t_{i-1},lT_p+t_i)$ with $T_i=t_i-t_{i-1}, i\in\mathcal{S}\triangleq\{1,2,\ldots,S\}, \Sigma_{i=1}^S T_i=T_p, t_0=0$ and $t_S=T_p$. For $t\in\mathcal{T}_i\triangleq[lT_p+t_{i-1},lT_p+t_i)$, the dynamics of ith subsystem is represented by the following LTV matrix functions:

$$\begin{cases}\mathcal{A}(t)=\mathcal{A}_i(t)=A_i+\sigma_i(t)\tilde{A}_i, \tilde{A}_i\triangleq A_{i+1}-A_i\\ \mathcal{B}(t)=\mathcal{B}_i(t)=B_i+\sigma_i(t)\tilde{B}_i, \tilde{B}_i\triangleq B_{i+1}-B_i\end{cases} \tag{2}$$

where LTV coefficients $\sigma_i(t)=\frac{t-lT_p-t_{i-1}}{T_i}\in[0,1), i\in\mathcal{S}$; $A_i\in\mathbb{R}^{n\times n}, B_i\in\mathbb{R}^{n\times n_u}, i\in\mathcal{S}$, are known constant matrices and $A_{S+1}=A_1, B_{S+1}=B_1$.

Remark 1: For $t\in\mathcal{T}_i, i\in\mathcal{S}$, if $\mathcal{A}_i(t)=\bar{A}_i, \mathcal{B}_i(t)=\bar{B}_i$, then PPTVS (1) will reduce to a PPLS, which has been studied in [9]. A sketch comparing the evolutions of subsystem matrices in PPTVS (1) and a PPLS is shown in Fig. 1.

In this article, one considers a periodic nonfragile time-varying control law

$$u(t)=(\mathcal{K}_i(t)+\Delta\mathcal{K}_i(t))\,x(t), t\in\mathcal{T}_i \tag{3}$$

and two types of controller perturbations as follows.

1) Norm-bounded additive perturbations

$$\Delta\mathcal{K}_i(t)=H_i\mathcal{F}_i(t)\mathcal{E}_i(t), t\in\mathcal{T}_i. \tag{4}$$

2) Norm-bounded multiplicative perturbations

$$\Delta\mathcal{K}_i(t)=H_i\mathcal{F}_i(t)\mathcal{E}_i(t)\mathcal{K}_i(t), t\in\mathcal{T}_i. \tag{5}$$

In (4) and (5), for $t\in\mathcal{T}_i$, $i\in\mathcal{S}$, consider time-varying parameter matrices

$$\mathcal{E}_i(t)=E_i+\sigma_i(t)\tilde{E}_i, \tilde{E}_i\triangleq E_{i+1}-E_i \tag{6}$$

where H_i and $E_i, i\in\mathcal{S}$, are known real constant matrices with appropriate dimensions; $\mathcal{F}_i(t), i\in\mathcal{S}$, are unknown functions that are continuous over the ith subinterval and right-continuous at the switching instants, satisfying

$$\mathcal{F}_i(t) \in \Phi_i \triangleq \left\{\mathcal{F}_i(t) \mid \mathcal{F}_i^T(t)\mathcal{F}_i(t) \le I\right\}, t \in \mathcal{T}_i, i \in \mathcal{S}. \quad (7)$$

Remark 2: In practice, the additive perturbations in (4) are usually due to the external disturbances and/or actuator perturbations affecting the controller gains in additive ways, such as variations or noises caused by electromagnetic fluctuations, changes in the voltage of actuator power, and varying currents in electronic components. On the other hand, the multiplicative perturbations in (5) describe the disturbances and/or perturbations with effects acting proportionally on the controller, which may be incurred by signal distortions.

Based on the previous study [18], the exponential stability characterized by a general convergence rate $\alpha^* > 0$ is applied to PPTVS (1) under the effects of controller perturbations.

Definition 1: PPTVS (1) with LTV subsystems in (2) and nonfragile control law (3) affected by perturbations (4) or (5) is α^*-exponentially stable if for all $\mathcal{F}_i(t), i \in \mathcal{S}$, which satisfy (7), there exist constants $\kappa \ge 1$ and $\alpha^* > 0$ such that the system solution from $x(0)$ satisfies $\|x(t)\| \le \kappa e^{-\alpha^* t}\|x(0)\|$ $\forall t \ge 0$.

Remark 3: The controller perturbations in (4) and (5) involve some partially known dynamics described by periodic piecewise time-varying matrices $\mathcal{E}_i(t), i \in \mathcal{S}$, which are different from traditional descriptions of parametric uncertainties in previous studies like [22]–[24], [33]. Such perturbations also bring more technical challenges to nonfragile controller design, since the time-varying dynamics in (2) and (4) or (5) can result in nonconvex variables that are difficult to decouple in closed-loop stability analysis.

III. Main Results

A. Closed-Loop Stability Analysis

For $t \ge 0$, consider a continuous T_p-periodic matrix function $\mathcal{P}(t) = \mathcal{P}_i(t) > 0, t \in \mathcal{T}_i$, satisfying $\lim_{t \to lT_p + t_i} \mathcal{P}(t) = \mathcal{P}(lT_p + t_i)$. For $t \in \mathcal{T}_i$, the upper right Dini derivative of continuous matrix function $\mathcal{P}(t)$ is given by

$$\mathcal{D}^+\mathcal{P}(t) = \mathcal{D}^+\mathcal{P}_i(t) = \limsup_{h \to 0^+} \frac{\mathcal{P}_i(t+h) - \mathcal{P}_i(t)}{h}. \quad (8)$$

Hence, for $x(t) \ne 0$, construct a continuous periodic quadratic Lyapunov function as

$$\mathcal{V}(t) = \mathcal{V}_i(t) = x^T(t)\mathcal{P}_i(t)x(t) > 0, t \in \mathcal{T}_i. \quad (9)$$

Based on Lyapunov function (9), a lemma on the general exponential stability of PPTVSs is given below.

Lemma 1 (see [18]*)* Consider PPTVS (1) with $u(t) = 0$. Given a scalar $\alpha^* > 0$, if there exist scalars $\alpha_i, i = 1, 2, \ldots, S$, and real symmetric T_p-periodic, continuous and Dini-differentiable matrix function $\mathcal{P}(t)$ defined on $t \in [0, \infty)$ such that, for $t \in \mathcal{T}_i$, $i \in \mathcal{S}$, $\mathcal{P}(t) = \mathcal{P}_i(t) > 0$, the following conditions hold:

$$\mathbf{sym}\left(\mathcal{P}_i(t)\mathcal{A}_i(t)\right) + \mathcal{D}^+\mathcal{P}_i(t) + \alpha_i\mathcal{P}_i(t) < 0 \quad (10)$$

$$2\alpha^* T_p - \sum_{i=1}^{S} \alpha_i T_i \le 0 \quad (11)$$

then the system is α^*-exponentially stable, that is, $\|x(t)\| \le \kappa e^{-\alpha^* t}\|x(0)\|$ $\forall t \ge 0$, where

$$\kappa = e^{\alpha^* T_p}\sqrt{\overline{\lambda}(\mathcal{P}(0))/\underline{\lambda}(\mathcal{P}(0))}\prod_{i=1}^{S}\max(1, e^{\hat{\mu}_i T_i}) \ge 1$$

and constant $\hat{\mu}_i$ satisfies $\hat{\mu}_i \ge \frac{1}{2}\overline{\lambda}(\mathcal{A}_i(t) + \mathcal{A}_i^T(t))$ for the ith subsystem, $t \in \mathcal{T}_i, i \in \mathcal{S}$.

Based on Lemma 1, a criterion of closed-loop stability is obtained for PPTVS (1) with nonfragile controller (3).

Theorem 1: Consider PPTVS (1) with nonfragile controller (3) and norm-bounded *additive perturbations* in (4). Given a scalar $\alpha^* > 0$, the closed-loop system is α^*-exponentially stable if there exist scalars $\xi_i > 0$, α_i, $i = 1, 2, \ldots, S$, and real symmetric T_p-periodic, continuous and Dini-differentiable matrix function $\mathcal{P}(t)$ defined on $t \in [0, \infty)$ such that, for $t \in \mathcal{T}_i, i \in \mathcal{S}$, $\mathcal{P}(t) = \mathcal{P}_i(t) > 0$, (11) and the following condition:

$$\begin{bmatrix} \mathcal{G}_i(t) & \xi_i\mathcal{P}_i(t)\mathcal{B}_i(t)H_i & \mathcal{E}_i^T(t) \\ * & -\xi_i I & 0 \\ * & * & -\xi_i I \end{bmatrix} < 0 \quad (12)$$

where

$$\begin{aligned}\mathcal{G}_i(t) = &\ \mathbf{sym}\left(\mathcal{P}_i(t)\mathcal{A}_i(t) + \mathcal{P}_i(t)\mathcal{B}_i(t)\mathcal{K}_i(t)\right) \\ &+ \mathcal{D}^+\mathcal{P}_i(t) + \alpha_i\mathcal{P}_i(t) \end{aligned} \quad (13)$$

hold for all $\mathcal{F}_i(t)$ satisfying (7).

Proof: For PPTVS (1) with nonfragile controller (3) and norm-bounded additive perturbations in (4), denote the closed-loop system as $\dot{x}(t) = \mathcal{A}_{ci}(t)x(t), t \in \mathcal{T}_i$, where $\mathcal{A}_{ci}(t) \triangleq \mathcal{A}_i(t) + \mathcal{B}_i(t)(\mathcal{K}_i(t) + \Delta\mathcal{K}_i(t)) = \mathcal{A}_i(t) + \mathcal{B}_i(t)(\mathcal{K}_i(t) + H_i\mathcal{F}_i(t)\mathcal{E}_i(t)), i \in \mathcal{S}$. With Lyapunov function (9), one has

$$\begin{aligned}&\mathcal{D}^+\mathcal{V}_i(t) + \alpha_i\mathcal{V}_i(t) \\ &= x^T(t)\left(\mathbf{sym}\left(\mathcal{P}_i(t)\mathcal{A}_{ci}(t)\right) + \mathcal{D}^+\mathcal{P}_i(t) + \alpha_i\mathcal{P}_i(t)\right)x(t) \\ &= x^T(t)\left(\mathcal{G}_i(t) + \mathbf{sym}\left(\mathcal{P}_i(t)\mathcal{B}_i(t)H_i\mathcal{F}_i(t)\mathcal{E}_i(t)\right)\right)x(t).\end{aligned}$$

From (12) and $\mathcal{G}_i(t) = \mathcal{G}_i^T(t), \xi_i > 0, i \in \mathcal{S}$, according to Schur complement equivalence, one has

$$\begin{aligned}&\mathcal{G}_i(t) + \xi_i\left(\mathcal{P}_i(t)\mathcal{B}_i(t)H_i\right)\left(\mathcal{P}_i(t)\mathcal{B}_i(t)H_i\right)^T \\ &+ \xi_i^{-1}\mathcal{E}_i^T(t)\mathcal{E}_i(t) < 0.\end{aligned} \quad (14)$$

For all $\mathcal{F}_i(t)$ satisfying (7), based on the fact

$$\begin{aligned}&\mathbf{sym}\left(\mathcal{P}_i(t)\mathcal{B}_i(t)H_i\mathcal{F}_i(t)\mathcal{E}_i(t)\right) \\ &\le \xi_i\left(\mathcal{P}_i(t)\mathcal{B}_i(t)H_i\right)\left(\mathcal{P}_i(t)\mathcal{B}_i(t)H_i\right)^T + \xi_i^{-1}\mathcal{E}_i^T(t)\mathcal{E}_i(t)\end{aligned}$$

one has

$$\begin{aligned}&\mathcal{G}_i(t) + \mathbf{sym}\left(\mathcal{P}_i(t)\mathcal{B}_i(t)H_i\mathcal{F}_i(t)\mathcal{E}_i(t)\right) \\ &\le \mathcal{G}_i(t) + \xi_i\left(\mathcal{P}_i(t)\mathcal{B}_i(t)H_i\right)\left(\mathcal{P}_i(t)\mathcal{B}_i(t)H_i\right)^T \\ &+ \xi_i^{-1}\mathcal{E}_i^T(t)\mathcal{E}_i(t) < 0\end{aligned} \quad (15)$$

which implies $\mathcal{D}^+\mathcal{V}_i(t) + \alpha_i\mathcal{V}_i(t) < 0, t \in \mathcal{T}_i$. Thus, when (11) and (12) hold, according to Lemma 1 and [31], the closed-loop PPTVS is α^*-exponentially stable. ■

Theorem 2: Consider PPTVS (1) with nonfragile controller (3) and norm-bounded *multiplicative perturbations* in (5). Given a scalar $\alpha^* > 0$, the closed-loop system is α^*-exponentially stable if there exist scalars $\xi_i > 0$, α_i, $i = 1, 2, \ldots, S$, and real symmetric T_p-periodic, continuous and Dini-differentiable matrix function $\mathcal{P}(t)$ defined on $t \in [0, \infty)$ such that, for $t \in \mathcal{T}_i, i = 1, 2, \ldots, S, \mathcal{P}(t) = \mathcal{P}_i(t) > 0$, (11) and the following condition:

$$\begin{bmatrix} \mathcal{G}_i(t) & \xi_i \mathcal{P}_i(t)\mathcal{B}_i(t)H_i & \mathcal{K}_i^T(t)\mathcal{E}_i^T(t) \\ * & -\xi_i I & 0 \\ * & * & -\xi_i I \end{bmatrix} < 0 \tag{16}$$

where $\mathcal{G}_i(t)$ is defined in (13), hold for all $\mathcal{F}_i(t)$ satisfying (7).

Proof: For PPTVS (1) with nonfragile controller (3) and norm-bounded multiplicative perturbations in (5), denote the closed-loop system as $\dot{x}(t) = \mathcal{A}_{ci}(t)x(t), t \in \mathcal{T}_i$, where $\mathcal{A}_{ci}(t) \triangleq \mathcal{A}_i(t) + \mathcal{B}_i(t)(I + H_i\mathcal{F}_i(t)\mathcal{E}_i(t))\mathcal{K}_i(t)$. When (11) and (16) hold, by Lyapunov function (9), Schur complement equivalence and following the procedures in the proof of Theorem 1, it is easy to prove that for all $\mathcal{F}_i(t)$ satisfying (7), $\mathcal{D}^+\mathcal{V}_i(t) + \alpha_i\mathcal{V}_i(t) < 0$, $t \in \mathcal{T}_i$. According to Lemma 1, the closed-loop PPTVS is α^*-exponentially stable. ∎

B. Nonfragile Controller Synthesis

To obtain the nonfragile controller gains by convex optimization, one divides each subinterval of PPTVS (1) into a number of small segments inspired by [13]. Taking interval $\mathcal{T}_i$, for example, with $M_i \in \mathbb{N}^+$, the interval is assumed to be divided into M_i segments with equal length $\delta_i = T_i/M_i$. Denoting $\theta_{i,m} \triangleq lT_p + t_{i-1} + m\delta_i$ for $m = 0, 1, \ldots, M_i - 1$, and $\mathcal{M}_i \triangleq \{0, 1, \ldots, M_i\}$, $i \in \mathcal{S}$, a continuous time-varying matrix function $\mathcal{P}(t)$ is defined by $\mathcal{P}(t) = \mathcal{P}_i(t), t \in \mathcal{T}_i$, where for $t \in [\theta_{i,m}, \theta_{i,m+1}) \in \mathcal{T}_i$

$$\mathcal{P}_i(t) = P_{i,m} + \varepsilon_{i,m}(t)\left(P_{i,m+1} - P_{i,m}\right) \tag{17}$$

with $P_{i,m} > 0, P_{i,M_i} = P_{i+1,0}, P_{S,M_i} = P_{1,0}, i \in \mathcal{S}, m \in \mathcal{M}_i$, and LTV coefficients

$$\varepsilon_{i,m}(t) = \frac{t - \theta_{i,m}}{\delta_i} = \frac{M_i(t - \theta_{i,m})}{T_i} \in [0, 1). \tag{18}$$

One can observe that periodic matrix function $\mathcal{P}(t) = \mathcal{P}(t + lT_p)$ is continuous at all the switching instants of time segments, which satisfies the requirements of $\mathcal{P}(t)$. Based on (8) and (17), $\mathcal{P}(t)$ is differentiable over each segmented time interval but not differentiable for all $t \geq 0$. Hence, one considers the upper right Dini derivative of $\mathcal{P}_i(t)$ for $t \in [\theta_{i,m}, \theta_{i,m+1})$, i.e.,

$$\mathcal{D}^+\mathcal{P}_i(t) = \frac{M_i(P_{i,m+1} - P_{i,m})}{T_i}. \tag{19}$$

According to the obtained theorems, it is worth noticing that a segment-based $\mathcal{P}(t)$ will lead to the coexistence of $\sigma_i(t)$ and $\varepsilon_{i,m}(t)$ in stability criteria, $m \in \mathcal{M}_i, i \in \mathcal{S}$. The following lemma is first proposed to facilitate the controller design.

Lemma 2: Let $f : [0, 1]^2 \to \mathbb{R}$ be a bounded matrix polynomial function defined as $f(\eta_1, \eta_2) = \Omega_0 + \eta_1\Omega_1 + \eta_2\Omega_2 + \eta_1\eta_2\Omega_{12}$, where scalars $\eta_1 \in [0, 1]$, $\eta_2 \in [0, 1]$, and $\Omega_0, \Omega_1, \Omega_2, \Omega_{12}$ are real symmetric matrices. Matrix polynomial $f(\eta_1, \eta_2) < 0$ if and only if

$$\Omega_0 < 0 \tag{20}$$

$$\Omega_0 + \Omega_1 < 0 \tag{21}$$

$$\Omega_0 + \Omega_2 < 0 \tag{22}$$

$$\Omega_0 + \Omega_1 + \Omega_2 + \Omega_{12} < 0. \tag{23}$$

Proof: Necessity: With $f(\eta_1, \eta_2) < 0$, $\eta_1 \in [0, 1]$ and $\eta_2 \in [0, 1]$, it is easy to obtain (20)–(23) by letting $(\eta_1, \eta_2) = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$.

Sufficiency: With $\Omega_0 < 0$, $\Omega_0 + \Omega_1 < 0$ and $\eta_1 \in [0, 1]$, one has

$$\Omega_0 + \eta_1\Omega_1 = (1 - \eta_1)\Omega_0 + \eta_1(\Omega_0 + \Omega_1) < 0. \tag{24}$$

Similarly, with $\Omega_0 + \Omega_2 < 0$, $\Omega_0 + \Omega_1 + \Omega_2 + \Omega_{12} < 0$ and $\eta_1 \in [0, 1]$, it follows that

$$\begin{aligned}&(\Omega_0 + \Omega_2) + \eta_1(\Omega_1 + \Omega_{12}) \\ &\quad = (1 - \eta_1)(\Omega_0 + \Omega_2) + \eta_1(\Omega_0 + \Omega_1 + \Omega_2 + \Omega_{12}) < 0\end{aligned}$$

which can be rewritten as

$$(\Omega_0 + \eta_1\Omega_1) + (\Omega_2 + \eta_1\Omega_{12}) < 0. \tag{25}$$

With $\eta_2 \in [0, 1]$, from (24) and (25), it holds that

$$\begin{aligned} f(\eta_1, \eta_2) &= (\Omega_0 + \eta_1\Omega_1) + \eta_2(\Omega_2 + \eta_1\Omega_{12}) \\ &= (1 - \eta_2)(\Omega_0 + \eta_1\Omega_1) \\ &\quad + \eta_2(\Omega_0 + \eta_1\Omega_1 + \Omega_2 + \eta_1\Omega_{12}) \\ &< 0. \end{aligned} \tag{26}$$

The proof is complete. ∎

Remark 4: The property in Lemma 2 can be schematically demonstrated by imposing $\Omega_0, \Omega_1, \Omega_2$, and Ω_{12} in $f(\eta_1, \eta_2)$ as scalars. For instance, let $f(\eta_1, \eta_2) = 1 - 2\eta_1 - 3\eta_2 + 5\eta_1\eta_2$ with scalar parameters satisfying Lemma 2. For $\eta_1 \in [0, 1], \eta_2 \in [0, 1]$, Fig. 2 shows the variations of $f(\eta_1, \eta_2)$, which can be found strictly less than zero. If one considers the boundary situation with $\Omega_0 + \Omega_1 + \Omega_2 + \Omega_{12} = 0$ by letting $f(\eta_1, \eta_2) = 1 - 2\eta_1 - 3\eta_2 + 6\eta_1\eta_2$, Fig. 3 shows that the bound of $f(\eta_1, \eta_2)$ will reach zero. If one considers $f(\eta_1, \eta_2) > 0$, by replacing

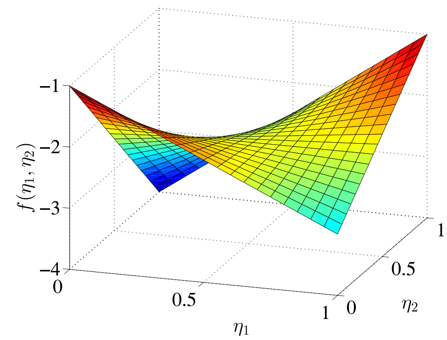

Fig. 2. Variations of function $f(\eta_1, \eta_2) = 1 - 2\eta_1 - 3\eta_2 + 5\eta_1\eta_2$.

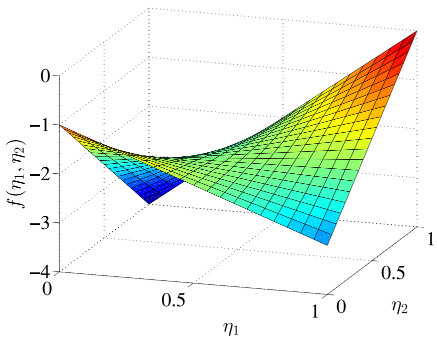

Fig. 3. Variations of function $f(\eta_1,\eta_2)=1-2\eta_1-3\eta_2+6\eta_1\eta_2$.

"$<$" to "$>$" in (20)–(23), the corresponding result is also an equivalent condition.

Using Lemma 2, two tractable criteria for nonfragile time-varying controller design are provided for PPTVSs.

Theorem 3: Consider PPTVS (1) with nonfragile controller (3) and norm-bounded additive perturbations in (4). Given a scalar $\alpha^*>0$, if there exist scalars $\xi_i>0$ and α_i, matrices $X_{i,m}>0$ and $Y_{i,m}$, $m\in\mathcal{M}_i$, $i\in\mathcal{S}$, (11) and the following conditions:

$$\Xi_{i,m,0}<0 \tag{27}$$

$$\Xi_{i,m,0}+\Xi_{i,m,1}<0 \tag{28}$$

$$\Xi_{i,m,0}+\Xi_{i,m,2}<0 \tag{29}$$

$$\Xi_{i,m,0}+\Xi_{i,m,1}+\Xi_{i,m,2}+\Xi_{i,m,3}<0 \tag{30}$$

$$X_{i,M_i}=X_{i+1,0},\, X_{S,M_i}=X_{1,0} \tag{31}$$

where

$$\Xi_{i,m,0}=\begin{bmatrix}\begin{matrix}\mathbf{sym}\left(A_iX_{i,m}+B_iY_{i,m}\right)\\ -\frac{M_i}{T_i}\tilde{X}_{i,m}+\alpha_iX_{i,m}\end{matrix} & \xi_iB_iH_i & X_{i,m}E_i^T\\ * & -\xi_iI & 0\\ * & * & -\xi_iI\end{bmatrix}$$

$$\Xi_{i,m,1}=\begin{bmatrix}\mathbf{sym}\left(\tilde{A}_iX_{i,m}+\tilde{B}_iY_{i,m}\right) & \xi_i\tilde{B}_iH_i & X_{i,m}\tilde{E}_i^T\\ * & 0 & 0\\ * & * & 0\end{bmatrix}$$

$$\Xi_{i,m,2}=\begin{bmatrix}\begin{matrix}\mathbf{sym}\left(A_i\tilde{X}_{i,m}+B_i\tilde{Y}_{i,m}\right)\\ +\alpha_i\tilde{X}_{i,m}\end{matrix} & 0 & \tilde{X}_{i,m}E_i^T\\ * & 0 & 0\\ * & * & 0\end{bmatrix}$$

$$\Xi_{i,m,3}=\begin{bmatrix}\mathbf{sym}\left(\tilde{A}_i\tilde{X}_{i,m}+\tilde{B}_i\tilde{Y}_{i,m}\right) & 0 & \tilde{X}_{i,m}\tilde{E}_i^T\\ * & 0 & 0\\ * & * & 0\end{bmatrix}$$

hold for all $\mathcal{F}_i(t)$ satisfying (7), then the closed-loop system is α^*-exponentially stable. The periodic nonfragile controller gains are given as

$$\mathcal{K}(t)=\mathcal{K}_i(t)=\mathcal{Y}_i(t)\mathcal{X}_i^{-1}(t), t\in\mathcal{T}_i \tag{32}$$

with time-varying matrix functions $\mathcal{X}_i(t)$ and $\mathcal{Y}_i(t)$ for $t\in[\theta_{i,m},\theta_{i,m+1})$ determined by

$$\mathcal{X}_i(t)=X_{i,m}+\varepsilon_{i,m}(t)\tilde{X}_{i,m} \tag{33}$$

$$\mathcal{Y}_i(t)=Y_{i,m}+\varepsilon_{i,m}(t)\tilde{Y}_{i,m} \tag{34}$$

where $\tilde{X}_{i,m}\triangleq X_{i,m+1}-X_{i,m}$, $\tilde{Y}_{i,m}\triangleq Y_{i,m+1}-Y_{i,m}$, $\varepsilon_{i,m}(t)=\frac{M_i(t-\theta_{i,m})}{T_i}\in[0,1)$, $m\in\mathcal{M}_i$, $i\in\mathcal{S}$.

Proof: For $t\geq 0$, consider a continuous periodic piecewise time-varying matrix function $\mathcal{X}(t)=\mathcal{X}_i(t)$, $t\in\mathcal{T}_i$, where for $\mathcal{X}_i(t)$ is in form of (33) with symmetric matrices $X_{i,m}>0$ satisfying (31) and $\tilde{X}_{i,m}=X_{i,m+1}-X_{i,m}$, $m\in\mathcal{M}_i$, $i\in\mathcal{S}$, and $\varepsilon_{i,m}(t)=\frac{M_i(t-\theta_{i,m})}{T_i}\in[0,1)$. Based on (8), the upper right Dini derivative of $\mathcal{X}_i(t)$ exists and is given by

$$\mathcal{D}^+\mathcal{X}_i(t)=\frac{M_i}{T_i}\tilde{X}_{i,m}, t\in[\theta_{i,m},\theta_{i,m+1}). \tag{35}$$

According to Lemma 2, conditions (27)–(34) imply that the following polynomial matrix inequality holds:

$$\Xi_{i,m,0}+\sigma_i(t)\Xi_{i,m,1}+\varepsilon_{i,m}(t)\Xi_{i,m,2}+\sigma_i(t)\varepsilon_{i,m}(t)\Xi_{i,m,3}<0 \tag{36}$$

which can be rewritten as

$$\begin{bmatrix}\begin{matrix}\mathbf{sym}\left(\mathcal{A}_i(t)\mathcal{X}_i(t)\right.\\ \left.+\mathcal{B}_i(t)\mathcal{Y}_i(t)\right)\\ -\mathcal{D}^+\mathcal{X}_i(t)+\alpha_i\mathcal{X}_i(t)\end{matrix} & \xi_i\mathcal{B}_i(t)H_i & \mathcal{X}_i(t)\mathcal{E}_i^T(t)\\ \xi_iH_i^T\mathcal{B}_i^T(t) & -\xi_iI & 0\\ \mathcal{E}_i(t)\mathcal{X}_i(t) & 0 & -\xi_iI\end{bmatrix}<0 \tag{37}$$

where

$$\begin{aligned}&\mathbf{sym}\left(\mathcal{A}_i(t)\mathcal{X}_i(t)+\mathcal{B}_i(t)\mathcal{Y}_i(t)\right)-\mathcal{D}^+\mathcal{X}_i(t)+\alpha_i\mathcal{X}_i(t)\\ &\quad=\mathbf{sym}\left(A_iX_{i,m}+B_iY_{i,m}\right)-\frac{M_i}{T_i}\tilde{X}_{i,m}+\alpha_iX_{i,m}\\ &\qquad+\sigma_i(t)\mathbf{sym}\left(\tilde{A}_iX_{i,m}+\tilde{B}_iY_{i,m}\right)\\ &\qquad+\varepsilon_{i,m}(t)\left(\mathbf{sym}\left(A_i\tilde{X}_{i,m}+B_i\tilde{Y}_{i,m}\right)+\alpha_i\tilde{X}_{i,m}\right)\\ &\qquad+\sigma_i(t)\varepsilon_{i,m}(t)\mathbf{sym}\left(\tilde{A}_i\tilde{X}_{i,m}+\tilde{B}_i\tilde{Y}_{i,m}\right)\end{aligned}$$

$$\xi_i\mathcal{B}_i(t)H_i=\xi_i(B_i+\sigma_i(t)\tilde{B}_i)H_i=\xi_iB_iH_i+\sigma_i(t)\xi_i\tilde{B}_iH_i,$$

$$\begin{aligned}\mathcal{X}_i(t)\mathcal{E}_i^T(t)&=X_{i,m}E_i^T+\sigma_i(t)X_{i,m}\tilde{E}_i^T+\varepsilon_{i,m}(t)\tilde{X}_{i,m}E_i^T\\ &\quad+\sigma_i(t)\varepsilon_{i,m}(t)\tilde{X}_{i,m}\tilde{E}_i^T.\end{aligned}$$

From $X_{i,m}>0$, $m\in\mathcal{M}_i$, $i\in\mathcal{S}$, one has $\mathcal{X}_i(t)>0$ and $\mathcal{X}_i^{-1}(t)>0$ for $t\geq 0$. Define a quadratic Lyapunov function $\mathcal{V}(t)=x^T(t)\mathcal{Q}(t)x(t)$, where continuous matrix function $\mathcal{Q}(t)=\mathcal{Q}_i(t)=\mathcal{X}_i^{-1}(t)$, $t\in\mathcal{T}_i$. Based on (32) and the fact that $\mathcal{D}^+\mathcal{X}_i^{-1}(t)=-\mathcal{X}_i^{-1}(t)\mathcal{D}^+\mathcal{X}_i(t)\mathcal{X}_i^{-1}(t)$, $t\in\mathcal{T}_i$, by multiplying both sides of inequality (37) with $\mathbf{diag}(\mathcal{Q}_i(t),I,I)$, one has

$$\begin{bmatrix}\begin{matrix}\mathbf{sym}\left(\mathcal{Q}_i(t)\mathcal{A}_i(t)\right.\\ \left.+\mathcal{Q}_i(t)\mathcal{B}_i(t)\mathcal{K}_i(t)\right)\\ +\mathcal{D}^+\mathcal{Q}_i(t)+\alpha_i\mathcal{Q}_i(t)\end{matrix} & \xi_i\mathcal{Q}_i(t)\mathcal{B}_i(t)H_i & \mathcal{E}_i^T(t)\\ \xi_iH_i^T\mathcal{B}_i^T(t)\mathcal{Q}_i(t) & -\xi_iI & 0\\ \mathcal{E}_i(t) & 0 & -\xi_iI\end{bmatrix}<0$$

which can be rewritten by (12). Thus, when conditions (11), (27)–(31) hold, from Theorem 1 one can conclude that the closed-loop PPTVS with nonfragile controller (3) and norm-bounded additive perturbations in (4) is α^*-exponentially stable for all $\mathcal{F}_i(t)$ satisfying (7). The proof is complete. ∎

Similarly, a sufficient condition of designing nonfragile controller that concerns multiplicative perturbations is provided in the following theorem.

Theorem 4: Consider PPTVS (1) with nonfragile controller (3) and norm-bounded multiplicative perturbations in (5). Given a scalar $\alpha^* > 0$, if there exist scalars $\xi_i > 0$ and α_i, matrices $X_{i,m} > 0$ and $Y_{i,m}$, $m \in \mathcal{M}_i$, $i \in \mathcal{S}$, (11) and the following conditions:

$$\Theta_{i,m,0} < 0 \tag{38}$$

$$\Theta_{i,m,0} + \Theta_{i,m,1} < 0 \tag{39}$$

$$\Theta_{i,m,0} + \Theta_{i,m,2} < 0 \tag{40}$$

$$\Theta_{i,m,0} + \Theta_{i,m,1} + \Theta_{i,m,2} + \Theta_{i,m,3} < 0 \tag{41}$$

$$X_{i,M_i} = X_{i+1,0}, X_{S,M_i} = X_{1,0} \tag{42}$$

where

$$\Theta_{i,m,0} = \begin{bmatrix} \begin{matrix}\mathbf{sym}\,(A_i X_{i,m} + B_i Y_{i,m}) \\ -\frac{M_i}{T_i}\tilde{X}_{i,m} + \alpha_i X_{i,m}\end{matrix} & \xi_i B_i H_i & Y_{i,m}^T E_i^T \\ * & -\xi_i I & 0 \\ * & * & -\xi_i I \end{bmatrix}$$

$$\Theta_{i,m,1} = \begin{bmatrix} \mathbf{sym}\left(\tilde{A}_i X_{i,m} + \tilde{B}_i Y_{i,m}\right) & \xi_i \tilde{B}_i H_i & Y_{i,m}^T \tilde{E}_i^T \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}$$

$$\Theta_{i,m,2} = \begin{bmatrix} \begin{matrix}\mathbf{sym}\left(A_i \tilde{X}_{i,m} + B_i \tilde{Y}_{i,m}\right) \\ +\alpha_i \tilde{X}_{i,m}\end{matrix} & 0 & \tilde{Y}_{i,m}^T E_i^T \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}$$

$$\Theta_{i,m,3} = \begin{bmatrix} \mathbf{sym}\left(\tilde{A}_i \tilde{X}_{i,m} + \tilde{B}_i \tilde{Y}_{i,m}\right) & 0 & \tilde{Y}_{i,m}^T \tilde{E}_i^T \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}$$

hold for all $\mathcal{F}_i(t)$ satisfying (7), then the closed-loop system is α^*-exponentially stable. The periodic nonfragile controller gains are obtained by (32)–(34).

Based on Lemma 2, one can prove Theorem 4 in a similar way by combining the relevant deductions in the proofs of Theorem 2 and Theorem 3. Hence, the proof is omitted.

In practice, there may be cases where the controller gains are more favorable to be continuous at all the switching instants. In this way, two continuous time-varying nonfragile control schemes are given in the following corollaries, which can be derived based on Theorem 3 and Theorem 4 by letting $Y_{i,1} = Y_i$, $Y_{i,2} = Y_{i+1}$, $i \in \mathcal{S}$.

Corollary 1: We consider PPTVS (1) with nonfragile controller (3) and norm-bounded additive perturbations in (4). Given a scalar $\alpha^* > 0$, if there exist scalars $\xi_i > 0$ and α_i, matrices $X_{i,m} > 0$ and $Y_{i,m}$, $m \in \mathcal{M}_i$, $i \in \mathcal{S}$, conditions (11), (27)–(31) and

$$Y_{i,M_i} = Y_{i+1,0}, Y_{S,M_i} = Y_{1,0} \tag{43}$$

hold for all $\mathcal{F}_i(t)$ satisfying (7), then the closed-loop system is α^*-exponentially stable. The periodic nonfragile controller gains are obtained from (32)–(34).

Corollary 2: Consider PPTVS (1) with nonfragile controller (3) and norm-bounded multiplicative perturbations in (5). Given a scalar $\alpha^* > 0$, if there exist scalars $\xi_i > 0$ and α_i, matrices $X_{i,m} > 0$ and $Y_{i,m}$, $m \in \mathcal{M}_i$, $i \in \mathcal{S}$, conditions (11), (38)–(42) and (43) hold for all $\mathcal{F}_i(t)$ satisfying (7), then the closed-loop system is α^*-exponentially stable. The periodic nonfragile controller gains are obtained from (32)–(34).

Compared with the previous studies [13], [31], it can be seen that without Lemma 2, it is difficult to derive the tractable criteria for controller design due to the different LTV coefficients $\sigma_i(t)$ and $\varepsilon_{i,m}(t)$ in Theorems 1 and 2, $i \in \mathcal{S}, m \in \mathcal{M}_i$. Using the selectable parameters $M_i, i \in \mathcal{S}$, the controller gains can become more flexible to increase the feasibility during the solution process via convex optimization. It should be noted that the number of conditions will be raised by larger values of $M_i, i \in \mathcal{S}$. To ensure a reasonable computational cost, M_i should be selected based on the balance of control effects and computing resources in practice.

Remark 5: Different from the corresponding constraints for PPLSs, the signs of $\alpha_i, i \in \mathcal{S}$, are independent of the stability of time-varying subsystems in PPTVSs, while one only needs to guarantee $\alpha^* > 0$ and (11). To ensure the convexity of the proposed criteria, $\alpha_i, i \in \mathcal{S}$, may be either given based on numerical experience, or iteratively generated and tested by some searching approaches like genetic algorithms [34].

Based on Theorems 3, 4 and Corollaries 1, 2, the proposed nonfragile controller design scheme is summarized in Algorithm 1, where Boolean operators flag_p and flag_K, respectively, indicate the type of perturbations (1 for additive, 0 for multiplicative) and the continuity of $\mathcal{K}(t)$ at the switching instants (1 for discontinuous, 0 for continuous), with default values as 1.

Algorithm 1: (Algorithm for Non-fragile Controller Design).

Input: n, n_u, T_p, S, T_i, A_i, B_i, E_i, H_i, M_i, α_i, $i \in \mathcal{S}$, α^*, flag_p, flag_K.
Output: $\xi_i > 0$, $X_{i,m} > 0$, $Y_{i,m}$, $m \in \mathcal{M}_i$, $i \in \mathcal{S}$.

```
Check the feasibility of condition (11). If TRUE,
continue. Otherwise, break and give warning.
if (flag_p, flag_K) = (1, 1) then
  Based on Theorem 3, solve (27)–(31).
else if (flag_p, flag_K) = (1, 0) then
  Based on Corollary 1, solve (27)–(31) and (43).
else if (flag_p, flag_K) = (0, 1) then
  Based on Theorem 4, solve (38)–(42).
else
  Based on Corollary 2, solve (38)–(42) and (43).
end if
Compute the controller gains using (32)–(34).
```

TABLE I
PARAMETER MATRICES AT THE SWITCHING INSTANTS

i	A_i	B_i	E_i
1	$\begin{bmatrix} 2 & 1 & 0 \\ 0.5 & 1 & 1 \\ 1 & 1.5 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.1 & 0.2 \\ 1 & 0.8 & 1 \end{bmatrix}$
2	$\begin{bmatrix} 1.5 & 1 & 0 \\ 0 & 0.6 & 1 \\ 0.5 & 1 & 0.9 \end{bmatrix}$	$\begin{bmatrix} 0.3 & 0 \\ 0 & 1 \\ 1 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 0.1 & 0.3 & 0.5 \\ 0.8 & 0.6 & 1.2 \end{bmatrix}$
3	$\begin{bmatrix} 1 & 1 & 0 \\ 0.5 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$	$\begin{bmatrix} 0.2 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.2 & 0 & 0.1 \\ 1.2 & -0.5 & 0.9 \end{bmatrix}$

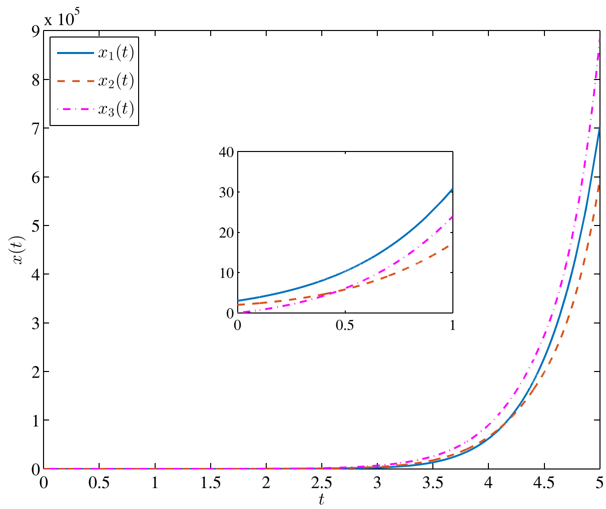

Fig. 4. Open-loop system state trajectory.

IV. CASE STUDIES

To validate the effectiveness of the proposed criteria, the nonfragile control of a numerical example and a mass-spring-damper benchmark system are considered in this section. Using Algorithm 1 with the MATLAB solver SeDuMi, the results obtained under different cases are compared to illustrate the effectiveness of the proposed control scheme.

A. Numerical Multi-Input Multi-Output (MIMO) PPTVS

First, one considers the nonfragile controller design for a MIMO PPTVS consisting of three LTV subsystems described by (2) with $T_p = 3.5$, $T_1 = 1$, $T_2 = 1.5$, and $T_3 = 1$ in appropriate time unit. Periodic time-varying matrix functions $\mathcal{A}(t)$, $\mathcal{B}(t)$, and $\mathcal{E}(t)$ are based on constant matrices $A_i, B_i, E_i, i = 1, 2, 3$, as given in Table I. We consider additive controller perturbations in (4), where $H_1 = I$, $H_2 = 1.5I$, $H_3 = 0.5I$, and uncertain continuous time-varying functions $\mathcal{F}_i(t) = \mathbf{diag}\big(a_1 \sin(2\pi t/T_p) + a_2 \sin(\pi t) + a_3 \sin(2\pi t), b_1 \cos(2\pi t/T_p) + b_2 \sin(\pi t) + b_3 \cos(2\pi t)\big)$ with randomly generated nonnegative scalars a_i, b_i, $i = 1, 2, 3$, satisfying $\sum_{i=1}^{3} a_i \leq 1$, $\sum_{i=1}^{3} b_i \leq 1$. With $x(0) = [3, 2, 0]^T$, the open-loop state trajectory is shown in Fig. 4, from which it can be seen that the open-loop PPTVS is unstable.

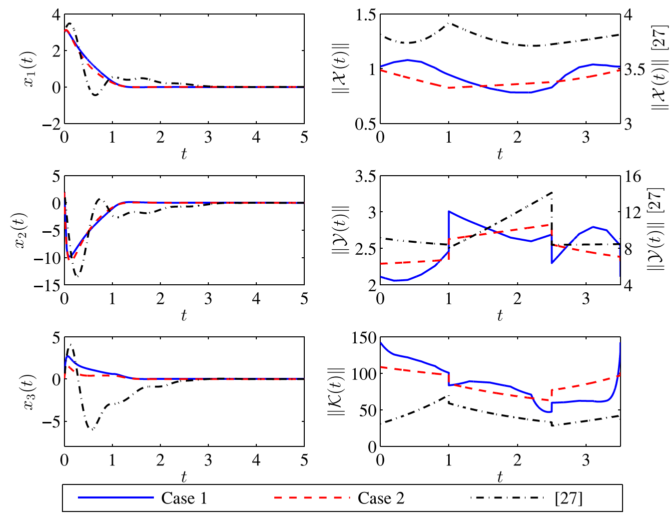

Fig. 5. Closed-loop system state trajectories and variations of $\|\mathcal{X}(t)\|$, $\|\mathcal{Y}(t)\|$, $\|\mathcal{K}(t)\|$ over one period for Case 1, Case 2 based on Theorem 3 and the method in [31].

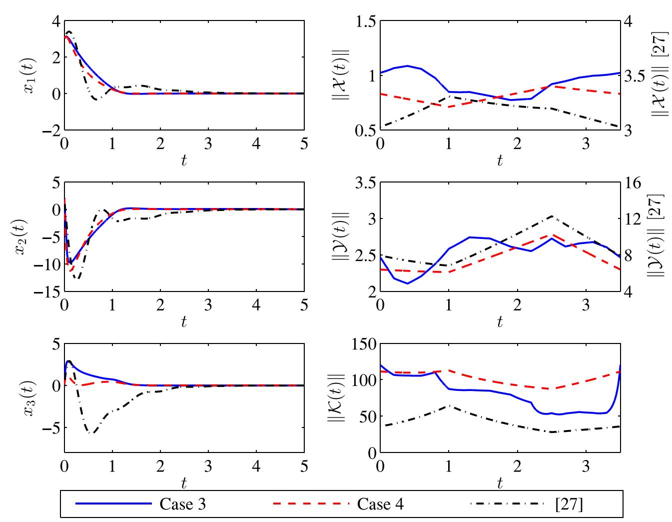

Fig. 6. Closed-loop system state trajectories and variations of $\|\mathcal{X}(t)\|$, $\|\mathcal{Y}(t)\|$, $\|\mathcal{K}(t)\|$ over one period for Case 3, Case 4 based on Corollary 1 and the method in [31].

Let $\alpha_1 = 0.8$, $\alpha_2 = 0.9$, $\alpha_3 = 1$, and $\alpha^* = \sum_{i=1}^{3} \alpha_i T_i / 2T_p = 0.45$. Consider the nonfragile controllers of the following cases ($i = 1, 2, 3$).

Case 1: $M_i = 5$ with discontinuous $\mathcal{Y}(t)$ and $\mathcal{K}(t)$.

Case 2: $M_i = 1$ with discontinuous $\mathcal{Y}(t)$ and $\mathcal{K}(t)$.

Case 3: $M_i = 5$ with continuous $\mathcal{Y}(t)$ and $\mathcal{K}(t)$.

Case 4: $M_i = 1$ with continuous $\mathcal{Y}(t)$ and $\mathcal{K}(t)$.

Based on Theorem 3 (for Case 1, Case 2) and Corollary 1 (for Case 3, Case 4), one can obtain the solutions of $X_{i,m}, Y_{i,m}$, $m \in \mathcal{M}_i$, $i = 1, 2, 3$. The time-varying controller gains can be computed via (32)–(34). The closed-loop system state trajectories and variations of $\|\mathcal{X}(t)\|$, $\|\mathcal{Y}(t)\|$, and $\|\mathcal{K}(t)\|$ over one period for the four cases are illustrated in Figs. 5 and 6 , where the continuity of $\mathcal{X}(t)$, $\mathcal{Y}(t)$, and $\mathcal{K}(t)$ at the switching instants

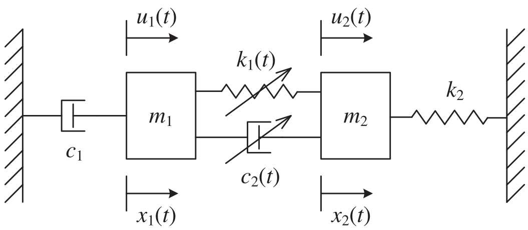

Fig. 7. Sketch of the considered mass-spring-damper system.

can be clearly reflected by the variations of their norms. For comparison, the closed-loop system state trajectories and the corresponding parameters obtained under the PPTVS controllers obtained by [31, Theorem 2 and Corollary 1] are also, respectively, shown in the figures.

With the same parameters of systems and $\alpha_i, i = 1, 2, 3$, from Figs. 5 and 6, it can be seen that the proposed nonfragile controllers can achieve better control effects and faster convergences against perturbations than those obtained by the existing methods for PPTVSs in [31]. On the other hand, benefiting from the divisions of subintervals during controller design, the differences in control effects of Theorem 3 and Corollary 1 are not obvious despite their different constraints in the continuity of $\mathcal{K}(t)$ at the switching instants. Moreover, from the variations of $\|\mathcal{X}(t)\|$, $\|\mathcal{Y}(t)\|$, and $\|\mathcal{K}(t)\|$ over one period of the four cases, one can observe that a larger M_i can result in more flexible time-varying controller gains especially for the cases imposing the continuity of controller gains at the switching instants, which explains the similar nonfragile control performances of the cases.

B. Mass-Spring-Damper System

In this example, the nonfragile controller design is considered for a mass-spring-damper system described by a PPTVS with four subsystems. The mass-spring-damper system can be used as a benchmark model for many practical vibration systems. For example, it provides a firm foundation for modeling some engineering objects with complex material properties like beams [35], which may exhibit time-periodic stiffness and/or damping properties due to periodic time-varying lengths [36] or time-periodic modulations [37]. Inspired by [36] and [38], the system is supposed to involve two masses, two spring elements, and two damping elements, as shown in Fig. 7, where some of the parameters are modulated to be time-periodic. The two system inputs are the force $u_1(t)$ applied to m_1 and the force $u_2(t)$ applied to m_2, while the displacement $x_1(t)$ of m_1 and the velocity $\dot{x}_2(t)$ of m_2 are chosen as the system outputs. The parameters of the system are given in Table II. For $t \geq 0$, based on the equations of motion, i.e.,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} c_1 + c_2(t) & -c_2(t) \\ -c_2(t) & c_2(t) \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} k_1(t) & -k_1(t) \\ -k_1(t) & k_1(t) + k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

TABLE II
PARAMETERS OF THE MASS-SPRING-DAMPER SYSTEM

Parameter	Meaning	Value	Unit
m_1	Mass of the left mass	10	kg
m_2	Mass of the right mass	4	kg
$k_1(t)$	Time-periodic stiffness between m_1 and m_2	[10, 15]	N/m
k_2	Constant stiffness between m_2 and the fixed world	5	N/m
c_1	Constant viscous damping between m_1 and the fixed world	0.2	N s/m
$c_2(t)$	Time-periodic viscous damping between m_1 and m_2	[0.2, 0.35]	N s/m

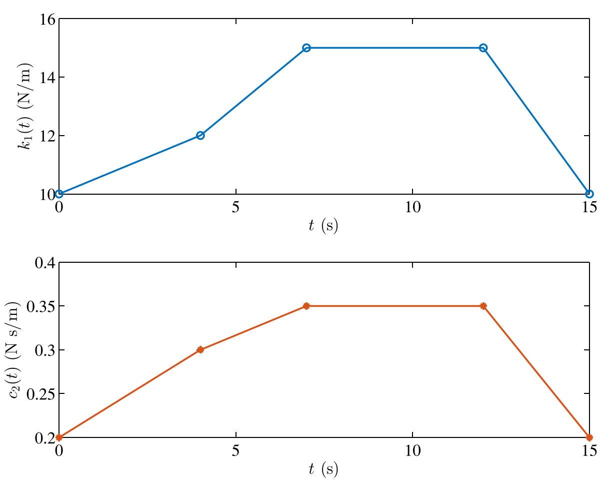

Fig. 8. $k_1(t)$ and $c_2(t)$ over one period.

A state-space representation of the system is derived as

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1(t)}{m_1} & \dfrac{k_1(t)}{m_1} & -\dfrac{c_1 + c_2(t)}{m_1} & \dfrac{c_2(t)}{m_1} \\ \dfrac{k_1(t)}{m_2} & -\dfrac{k_1(t) + k_2}{m_2} & \dfrac{c_2(t)}{m_2} & -\dfrac{c_2(t)}{m_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & \dfrac{1}{m_1} & 0 \\ 0 & 0 & 0 & \dfrac{1}{m_2} \end{bmatrix}^T u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) \tag{44}$$

with state vector $x(t) = [x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)]^T$, control input vector $u(t) = [u_1(t), u_2(t)]^T$, output vector $y(t) = [x_1(t), \dot{x}_2(t)]^T$, and periodic piecewise time-varying $k_1(t) = k_1(t + lT_p)$, $c_2(t) = c_2(t + lT_p)$, $l = 0, 1, \ldots$. Consider a fundamental period $T_p = 15$ s with $T_1 = 4$ s, $T_2 = 3$ s, $T_3 = 5$ s, and $T_4 = 3$ s, the variations of $k_1(t)$ and $c_2(t)$ over one period are shown in Fig. 8. Hence, system (44) is a PPTVS. The objective

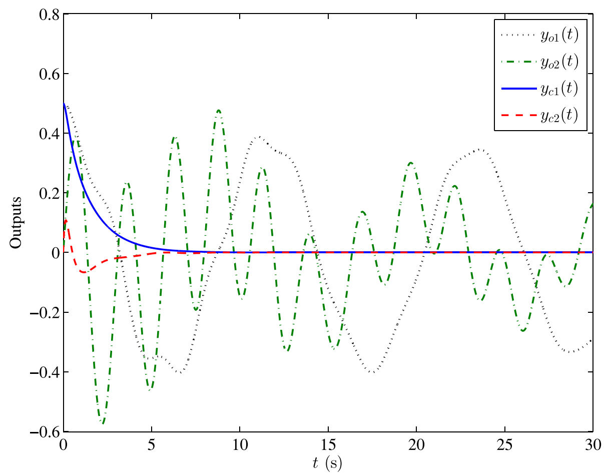

Fig. 9. Open-loop and closed-loop system outputs.

here is to design a nonfragile controller for vibration attenuation of the masses.

For PPTVS (44), consider $M_1 = 4$, $M_2 = 3$, $M_3 = 5$, $M_4 = 3$, $\alpha_i = 0.5$, $i = 1, 2, 3, 4$, $\alpha^* = \sum_{i=1}^{4} \alpha_i T_i / 2T_p = 0.25$ and multiplicative controller perturbations in (5), where for $i = 1, 2, 3, 4$, $H_i = 1.2I$, $\mathcal{F}_i(t) = \mathbf{diag}(\sin(t), 0.5\sin(t))$ and $\mathcal{E}_i(t)$ satisfying (6) with

$$E_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0.1 \end{bmatrix}, E_2 = \begin{bmatrix} 0.1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}, E_4 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}.$$

For convenience of application, a nonfragile control scheme with periodic time-varying controller gains continuous at the switching instants is obtained by Corollary 2, which is developed from Theorem 4. One obtains $\xi_1 = 1.8$, $\xi_2 = 1.8452$, $\xi_3 = 1.496$, $\xi_4 = 1.6729$ and the relevant matrix solutions. Denote $y_o(t) = [y_{o1}(t), y_{o2}(t)]^T = [x_{o1}(t), \dot{x}_{o2}(t)]^T$ and $y_c(t) = [y_{c1}(t), y_{c2}(t)]^T = [x_{c1}(t), \dot{x}_{c2}(t)]^T$ as the open-loop system output and the closed-loop system output, respectively, which are obtained under the designed controller with multiplicative perturbations and $x(0) = [0.5, 0.1, 0, 0]^T$. Fig. 9 shows the variations of open-loop and closed-loop system outputs. It can be seen that the closed-loop system is stable against multiplicative controller perturbations, and the concerned vibrations are attenuated. Therefore, the advantages of the proposed control scheme can be summarized as the superior technical flexibility in controller than previous studies, as well as the effectiveness in achieving resilience and fast convergence under controller perturbations.

V. Conclusion

This article developed the nonfragile controller design scheme for a class of PPTVSs with LTV subsystems using a new matrix polynomial lemma. Additive and multiplicative perturbations, which can be partially characterized by periodic piecewise time-varying parameters, were considered. The proposed lemma provided an alternative approach to make the nonfragile controller design of PPTVSs amenable to convex optimization, and the controller gains can be more flexible due to some selectable parameters of subinterval division. Sufficient conditions on stability analysis and tractable nonfragile controller synthesis were established based on a periodic Lyapunov function in time interpolative form. Case studies on a numerical MIMO PPTVS model and a mass-spring-damper system demonstrated the effectiveness of the designed controllers. Compared with the existing control method of PPTVSs, the proposed nonfragile control scheme not only performs well in guaranteeing the stability, but also can achieve faster convergence under the impacts of uncertain controller perturbations. In future work, research efforts will be extended to the input–output performance analysis, especially for the cases involving more complicated periodic time-varying uncertainties, faults, and industry-oriented requirements [39], [40].

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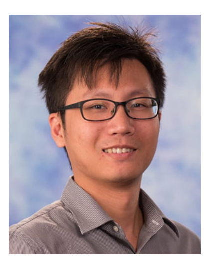

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